Mathematica Moravica
Vol. **17-2** (2013), 51–61

Pseudo-Slant Warped Product Submanifolds of a Kenmotsu Manifold

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Abstract. In this paper we study pseudo-slant warped product submanifolds of a Kenmotsu manifold. We obtain some basic results in this setting and prove an inequality for squared norm of second fundamental form and equality case is also discussed. Finally, we also give examples of these submanifolds.

1. Introduction

In [15] S. Tanno classified the connected almost contact metric manifold whose automorphism group has maximum dimension, there are three classes

(a) Homogeneous normal contact Riemannian manifolds with constant ϕ holomorphic sectional curvature if the sectional curvature of the plane section containing ξ, say $C(X,\xi) > 0$.
(b) Global Riemannian product of a line or a circle and Kaehlerian manifold with constant holomorphic sectional curvature, $C(X,\xi) = 0$.
(c) A warped product space $R \times_f C^n$, if $C(X,\xi) < 0$.

Manifolds of class (a) are characterized by some tensorial equations, it has a Sasakian structure and manifolds of class (b) are characterized by some tensor equations called Cosymplectic manifolds. Kenmotsu [9] obtained some tensorial equations to Characterize manifolds of class (c), these manifolds are called Kenmotsu manifolds.

The study of slant immersions in almost Hermitian manifolds was initiated by B. Y.Chen [4]. A. Lotta [1] extended the notion of slant immersions in the setting of almost contact metric manifolds. N. Papaghiuc [12] introduced a class of submanifolds in an almost Hermitian manifold, called the semi-slant submanifolds , this class includes the class of proper CR-submanifolds and slant submanifolds. J.L. Cabrerizo et al. [8] initiated study of contact version of semi-slant submanifolds and also defined bi-slant submanifolds. A step forward A. Carriazo [2] defined and study bi-slant immersions in

2010 *Mathematics Subject Classification.* 53C25, 53C40, 53C42, 53D15.
Key words and phrases. Warped product, Pseudo-slant, Kenmotsu manifold.

almost Hermitian manifolds and simultaneously gave the notion of anti-slant submanifolds in almost Hermitian manifolds. After that V.A. Khan et al. [17] renamed these submanifolds as Pseudo-slant submanifolds and studied in the setting of Sasakian manifolds. P seudo-slant submanifolds includes the class of semi-invariant and slant submanifolds.

R.L. Bishop and B. O' Neil [13] introduced the notion of warped product manifolds. These manifolds are generalization of Riemannian product manifolds and occur naturally. Recently, many important physical applications of warped product manifolds have been discovered, giving motivation to study of these spaces with differential geometric point of view. For instance, it has been accomplished that warped product manifolds provide an excellent setting to model space time near black hole or bodies with large gravitational fields (c.f., [6, 13, 14]). In this paper we study non-trivial warped product submanifolds of a Kenmotsu manifold and in this study there emerges natural problems of finding the estimates of the squared norm of second fundamental form. This study predict the geometric behavior of underlying warped product submanifolds. Further, as it is known that the warping function of a warped product manifold is a solution of some partial differential equations (c.f., [3]) and most of physical phenomenon are described by partial differential equations. We hope that our study may find applications in Physics as well as in Engineering.

2. Preliminaries

A $2n+1$ dimensional C^∞ manifold $\bar{M}$ is said to have an almost contact structure if there exist on M a tensor field ϕ of type $(1,1)$, a vector field ξ and 1-form η satisfying the following properties

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1. \tag{2.1}$$

There always exists a Riemannian metric g on an almost contact manifold $\bar{M}$ satisfying the following conditions

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi), \tag{2.2}$$

where X, Y are vector fields on $\bar{M}$.

An almost contact metric structure (ϕ, ξ, η, g) is said to be Kenmotsu manifold, if it satisfies the following tensorial equation [9]

$$(\bar{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.3}$$

for any $X, Y \in T\bar{M}$, where $T\bar{M}$ is the tangent bundle of $\bar{M}$ and $\bar{\nabla}$ denotes the Riemannian connection of the metric g. Moreover, for a Kenmotsu manifold

$$\bar{\nabla}_X \xi = X - \eta(X)\xi. \tag{2.4}$$

Let M be a submanifold of an almost contact metric manifold $\bar{M}$ with induced metric g and if ∇ and $\nabla^{\perp}$ are the induced connection on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively then Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.5}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.6}$$

for each $X, Y \in TM$ and $N \in T^{\perp}M$, where h and A_N are the second fundamental form and the shape operator respectively for the immersion of M into $\bar{M}$ and they are related as

$$g(h(X,Y),N) = g(A_N X, Y), \tag{2.7}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as on M.

For any $X \in TM$, we write

$$\phi X = PX + FX, \tag{2.8}$$

where PX is the tangential component and FX is the normal component of ϕX.

Similarly, for any $N \in T^{\perp}M$, we write

$$\phi N = tN + fN, \tag{2.9}$$

where tN is the tangential component and fN is the normal component of ϕN. The covariant derivatives of the tensor field P and F are defined as

$$(\bar{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y, \tag{2.10}$$

$$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y. \tag{2.11}$$

From equations (2.3), (2.5), (2.6), (2.8) and (2.9) we have

$$(\bar{\nabla}_X P)Y = A_{FY}X + th(X,Y) - g(X,PY)\xi - \eta(Y)PX \tag{2.12}$$

$$(\bar{\nabla}_X F)Y = fh(X,Y) - h(X,PY) - \eta(Y)FX. \tag{2.13}$$

Definition 2.1 ([1])**.** A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be slant submanifold if for any $x \in M$ and $X \in T_xM - \langle \xi \rangle$ the angle between X and ϕX is constant. The constant angle $\theta \in [0, \pi/2]$ is then called slant angle of M in $\bar{M}$. If $\theta = 0$ the submanifold is invariant submanifold, if $\theta = \pi/2$ then it is anti-invariant submanifold, if $\theta \neq 0, \pi/2$ then it is proper slant submanifold.

For slant submanifolds of contact manifolds J.L. Cabrerizo et al. [8] proved the following Lemma.

Lemma 2.1. *Let M be a submanifold of an almost contact manifold $\bar{M}$, such that $\xi \in TM$ then M is slant submanifold if and only if there exist a constant $\lambda \in [0,1]$ such that*

$$P^2 = -\lambda(I - \eta \otimes \xi), \tag{2.14}$$

where $\lambda = \cos^2\theta$.

Thus, one has the following consequences of above formulae

$$g(PX, PY) = \cos^2\theta[g(X, Y) - \eta(X)\eta(Y)], \tag{2.15}$$

$$g(FX, FY) = \sin^2\theta[g(X, Y) - \eta(X)\eta(Y)]. \tag{2.16}$$

A submanifold M of $\bar{M}$ is said to be pseudo-slant submanifold of an almost contact manifold $\bar{M}$ if there exist two orthogonal complementary distributions $D^{\perp}$ and D_θ on M such that

(i) $TM = D^{\perp} \oplus D_\theta \oplus \langle\xi\rangle$,
(ii) The distribution $D^{\perp}$ is anti-invariant i.e., $\phi D^{\perp} \subseteq T^{\perp}M$,
(iii) The distribution D_θ is slant with slant angle $\theta \neq \pi/2$.

It is straight forward to see that semi-invariant submanifolds and slant submanifolds are Pseudo-slant submanifolds with $\theta = 0$ and $D^{\perp} = \{0\}$, respectively.

If μ is invariant subspace under ϕ of the normal bundle $T^{\perp}M$, then in the case of pseudo-slant submanifold, the normal bundle $T^{\perp}M$ can be decomposed as

$$T^{\perp}M = \mu \oplus FD_\theta \oplus FD^{\perp}. \tag{2.17}$$

A pseudo-slant submanifold M is called a pseudo-slant product if the distributions $D^{\perp}$ and D_θ are involutive and parallel on M. In this case M is foliated by the leaves of these distributions.

As a generalization of the product manifolds and in particular of a pseudo-slant product submanifold, one can consider warped product of manifolds which are defined as

Definition 2.2. Let (B, g_B) and (F, g_F) be two Riemannian manifolds with Riemannian metric g_B and g_F respectively and f be a positive differentiable function on B. The warped product of B and F is the Riemannian manifold $(B \times F, g)$, where

$$g = g_B + f^2 g_F.$$

For a warped product manifold $N_1 \times_f N_2$, we denote by D_1 and D_2 the distributions defined by the vectors tangent to the leaves and fibers, respectively. In other words, D_1 is obtained by the tangent vectors of N_1 via the horizontal lift and D_2 is obtained by the tangent vectors of N_2 via vertical lift. In case of pseudo-slant warped product submanifolds D_1 and D_2 are replaced by $D^{\perp}$ and D_θ, respectively.

The warped product manifold $(B \times F, g)$ is denoted by $B \times_f F$. If X is the tangent vector field to $M = B \times_f F$ at (p, q) then

$$\|X\|^2 = \|d\pi_1 X\|^2 + f^2(p)\|d\pi_2 X\|^2.$$

R.L. Bishop and B. O'Neill [13] proved the following

Theorem 2.1. *Let $M = B \times_f F$ be warped product manifolds. If $X, Y \in TB$ and $V, W \in TF$ then*

(i) $\nabla_X Y \in TB$,
(ii) $\nabla_X V = \nabla_V X = (\frac{Xf}{f})V$,
(iii) $\nabla_V W = \frac{-g(V,W)}{f}\nabla f$.

∇f is the gradient of f and is defined as

$$g(\nabla f, X) = Xf, \tag{2.18}$$

for all $X \in TM$.

Corollary 2.1. *On a warped product manifold $M = N_1 \times_f N_2$, the following statements hold*

(i) *N_1 is totally geodesic in M,*
(ii) *N_2 is totally umbilical in M.*

Throughout, we denote by $N_\perp$ and N_θ an anti-invariant and a slant submanifold respectively of an almost contact metric manifold $\bar{M}$.

K.A. Khan et al. [10] proved the following Corollary

Corollary 2.2. *Let $\bar{M}$ be a Kenmotsu manifold and N_1 and N_2 be any Riemannian submanifolds of $\bar{M}$, then there do not exist a warped product submanifold $M = N_1 \times_f N_2$ of $\bar{M}$ such that ξ is tangential to N_2.*

Thus, we assume that the structure vector field ξ is tangential to N_1 of a warped product submanifold $N_1 \times_f N_2$ of $\bar{M}$.

In this paper we will consider the warped product of the type $N_\theta \times_f N_\perp$ and $N_\perp \times_f N_\theta$. The warped product of the type $N_\theta \times_f N_\perp$ is called warped product semi-slant submanifolds these type of warped product studied by M. Atceken [11], they proved that the warped product $N_\theta \times_f N_\perp$ does not exist. The warped product of the type $N_\perp \times_f N_\theta$ is called pseudo-slant warped product these submanifolds were studied by Siraj Uddin et. al. [16]. In this paper we will study the warped product of type $N_\perp \times_f N_\theta$.

3. Pseudo-slant warped product submanifolds

Throughout this section we will study the warped product of the type $N_\perp \times_f N_\theta$ for these submanifolds by Theorem 2.1 we have

$$\nabla_X Z = \nabla_Z X = Z lnf X, \tag{3.1}$$

for any $X \in TN_\theta$ and $Z \in TN_\perp$.

Now we start with the some properties of second fundamental form

Proposition 3.1. *Let $M = N_\perp \times_f N_\theta$ be a pseudo-slant warped product submanifolds of a Kenmotsu manifold $\bar{M}$, then*

(i) $g(h(X,Z),FY) = g(X,Y)g(H,FZ) + g(X,PY)(\eta(Z) - Zlnf)$,
(ii) $g(h(Z,Z),FW) = g(h(Z,W),FZ)$,
(iii) $g(h(Z,W),FX) - g(h(X,Z),FW) = 0$,

for any $X \in TN_\theta$ and $Z, W \in TN_\perp$.

Proof. For any $X, Y \in TN_\theta$ by equation (2.12), we have

$$\nabla_X PY - P\nabla_X Y = A_{FY}X + th(X,Y) - g(X,PY)\xi,$$

taking inner product with $Z \in TN_\perp$ and using equation (3.1), we get

$$g(h(X,Z),FY) - g(h(X,Y),FZ) = g(X,PY)(\eta(Z) - Zlnf). \tag{3.2}$$

Since N_θ is totally umbilical the above equation yields

$$g(h(X,Z),FY) = g(X,Y)g(H,FZ) + g(X,PY)(\eta(Z) - Zlnf),$$

which proves the part (i).

As $N_\perp$ is totally geodesic, then from equation (2.12), we have

$$A_{FW}Z + th(Z,W) - \eta(W)PZ = 0,$$

taking inner product with $Z \in TN_\perp$, the above equation gives

$$g(h(Z,Z),FW) = g(h(Z,W),FZ),$$

which is the part (ii) of proposition.

Again by equations (2.12) and (3.1), we have

$$A_{FX}Z + th(X,Z) = 0,$$

taking inner product with $W \in TN_\perp$, we get part (iii) of Proposition. □

Corollary 3.1. *Let $M = N_\perp \times_f N_\theta$, then the following are equivalent*

(i) $h(X,Z) \in \mu \oplus FD^\perp$,
(ii) $H \in \mu \oplus FD_\theta$,

for any $X \in TN_\theta$ and $Z \in TN_\perp$

Proof. From part (i) of Proposition 3.1, we have

$$g(h(X,Z),FX) = g(H,FZ)\|X\|^2,$$

for any $X \in TN_\theta$ and $Z \in TN_\perp$, which proves the Corollary. □

Theorem 3.1. *Let $M = N_\perp \times_f N_\theta$ be a pseudo-slant warped product submanifold of a Kenmotsu manifold, then*

$$g(h(PX,Z),FX) = (\eta(Z) - Zlnf), \tag{3.3}$$

for any $X \in TN_\theta$ and $Z \in TN_\perp$.

Proof. By equations (2.12) and (3.1)

$$-ZlnfPX = A_{FZ}X + th(X,Z) - \eta(Z)PX,$$

taking inner product with PX and using equation (2.15), above equation gives

$$-\cos^2\theta Zlnf\|X\|^2 = g(h(X,PX),FZ) - g(h(X,Z),FPX) - \eta(Z)\cos^2\theta\|X\|^2.$$

Since N_θ is totally umbilical, then above equation reduced to

$$g(h(X,Z),FPX) = \cos^2\theta(Zlnf - \eta(Z)), \tag{3.4}$$

replacing X by PX and using equation (2.15), we get equation (3.3). □

Corollary 3.2. *Let $M = N_\perp \times_f N_\theta$ be a pseudo-slant warped product of a Kenmotsu manifold if $H \in \mu \oplus FD_\theta$, then*

$$Zlnf = \eta(Z),$$

for any $Z \in TN_\perp$.

The Corollary 3.2 follows from Corollary 3.1 and equation (3.4).

Note 3.1. From equations (2.4), (2.8) and (3.3) we have

$$\xi lnf = 1,$$

by equation (2.18), it is easy to see that

$$\nabla lnf = \xi,$$

where ∇lnf denotes the gradient of lnf and above equation can also be written as

$$\sum_{i=0}^{p} \frac{\partial lnf}{\partial x_i} = \xi, \tag{3.5}$$

where $p+1$ is the dimension of $N_\perp$, equation (3.5) is the first order partial differential equation and has a unique solution i.e., pseudo-slant warped product submanifolds exist in this setting.

Let us denote by $D_\perp$ and D_θ the tangent bundles on $N_\perp$ and N_θ respectively and let $\{Z_0 = \xi, Z_1, \ldots Z_p\}$ and $\{X_1, \ldots, X_q, X_{q+1} = PX_1, \ldots, X_{2q} = PX_q\}$ be local orthonormal frames of vector fields on $N_\perp$ and N_θ respectively with $p+1$ and $2q$ real dimension, since $h(X,\xi) = 0$ for all $X \in TM$, then the second fundamental form can be written as

$$\|h\|^2 = \sum_{i,j=1}^{2p} g(h(X_i,X_j),h(X_i,X_j))+$$
$$\sum_{i=1}^{2p}\sum_{r=1}^{q}(g(h(X_i,Z_r),h(X_i,Z_r))+ \tag{3.6}$$
$$\sum_{r,s=1}^{q} g(h(Z_r,Z_s),h(Z_r,Z_s)).$$

Theorem 3.2. *Let $M = N_\perp \times_f N_\theta$ be a pseudo-slant warped product submanifold of a Kenmotsu manifold $\bar{M}$ with $N_\perp$ and N_θ anti-invariant and slant submanifolds respectively of $\bar{M}$. If $\eta(Z) \geq 2Zlnf$ for all $Z \in TN_\perp$, then*

(i) *The squared norm of the second fundamental form h satisfies*

$$\|h\|^2 \geq q\csc^2\theta\{3+\cos^4\theta\}\|\nabla lnf\|^2, \tag{3.7}$$

where ∇lnf is the gradient of lnf and $2q$ is the dimension N_θ.

(ii) *The equality holds if $h(D^\perp,D^\perp) = 0$, $h(D_\theta,D_\theta) = 0$, $h(PX,Z)$ and $h(X,Z)$ are orthogonal to FX and FPX for all $X \in TN_\theta$ and $Z \in TN_\perp$ and $\eta(Z) = 2ZlnZ$.*

Proof. In view of the decomposition (2.17), we may write

$$h(U,V) = h_{FD_\theta}(U,V) + h_{FD^\perp}(U,V) + h_\mu(U,V), \tag{3.8}$$

for each $U,V \in TM$, where $h_{FD_\theta}(U,V) \in FD_\theta$, $h_{FD_\perp}(U,V) \in FD^\perp$ and $h_\mu(U,V) \in \mu$, with

$$h_{FD_\theta}(U,V) = \sum_{i=1}^{2q} h^i(U,V)FX_i \tag{3.9}$$

$$h^i(U,V) = \csc^2\theta g(h(U,V),FX_i) \tag{3.10}$$

for each $U,V \in TM$. In view of above formulae we have

$$g(h_{FD_\theta}(PX_i,Z_r),h_{FD_\theta}(PX_i,Z_r)) = h^i(PX_i,Z_r)g(FX_i,h(PX_i,Z_r)) + \sum_{s\neq i}(h^s(PX_i,Z_r))^2 g(FX_i,FX_i).$$

Now using equations (3.3) and (2.15), the above equation becomes

$$g(h_{FD_\theta}(PX_i,Z_r),h_{FD_\theta}(PX_i,Z_r)) = \csc^2\theta(\eta(Z_r) - Z_r lnf)^2 + \sin^2\theta\sum_{s\neq i}(h^s(PX_i,Z_r))^2.$$

Summing over $i = 1, \ldots, 2q$ and $r = 1, \ldots, p$ and using the assumption $\eta(Z) \geq 2Zlnf$, we get the following inequality

$$g(h_{FD_\theta}(PX_i, Z_r), h_{FD_\theta}(PX_i, Z_r)) \geq 2q \csc^2\theta \|\nabla lnf\|^2 + \sin^2\theta \sum_{s\neq i}(h^s(PX_i, Z_r))^2. \tag{3.11}$$

Since we have choose the orthonormal frame of D_θ as $\{X_1, \ldots, X_q, X_{q+1} = PX_1, \ldots, X_{2q} = PX_q\}$, then the second term in above inequality can be written as

$$\sin^2 \csc^4\theta \sum_{r=1}^{p}\Bigg[\sum_{i=1}^{q}\Big\{(g(h(X_i, Z_r), FPX_i))^2 + (g(h(PX_i, Z_r), FX_i))^2\Big\} + \sum_{i=1}^{q}\sum_{s=1,\, s\neq i}^{q}\Big\{(g(h(X_i, Z_r), FPX_s))^2 + (g(h(PX_i, PZ_r), FX_s))^2\Big\}\Bigg].$$

The first two terms in view of equations (3.3) and (3.4) can be written as

$$\csc^2\theta \sum_{r=1}^{p}\left[\sum_{i=1}^{q}\left\{\cos^4\theta(Zlnf - \eta(Z))^2 + (Zlnf - \eta(Z))^2\right\}\right].$$

In view of assumption $\eta(Z) \geq 2Zlnf$ the above expression is greater than equal to following

$$\csc^2[q\cos^4\theta\|\nabla lnf\|^2 + q\|\nabla lnf\|^2]. \tag{3.12}$$

From inequalities (3.11) and (3.12), we get

$$g(h_{FD_\theta}(PX_i, Z_r), h_{FD_\theta}(PX_i, Z_r)) \geq q\csc^2\theta\{3 + \cos^4\theta\}\|\nabla lnf\|^2. \tag{3.13}$$

The inequality (3.7) follows from (3.6) and (3.13). □

Now we have the following examples for $N_\perp \times_f N_\theta$.

Example 3.1. Consider the complex space C^4 with the usual Kaehler structure and real global coordinates $(x^1, y^1, x^2, y^2, x^3, y^3, x^4, y^4)$. Let $\bar{M} = R \times_f C^4$ be the warped product between the real line R and C^4, where warping function is e^t and t being the global coordinates in R, then $\bar{M}$ is a Kenmotsu manifold. Now defining the orthogonal basis

$$e_1 = \partial/\partial x_1, \qquad e_2 = \sin\theta\partial/\partial x_4 - \cos\theta\partial/\partial y_4,$$
$$e_3 = -\cos\theta\partial/\partial x_4 - \sin\theta\partial/\partial y_4, \qquad e_4 = \partial/\partial t.$$

Obviously the distributions $D_\theta = \langle e_2, e_3\rangle$ and $D_\perp = \langle e_1, e_4\rangle$ and denoted by N_θ and $N_\perp$, then $N_\perp \times_f N_\theta$ is a pseudo-slant warped product submanifold isometrically immersed in $\bar{M}$, here the warping function is $f = e^t$.

In general, we have

Example 3.2. Consider the complex space C^{2n} with the usual Kaehler structure and real global coordinates $(x^1, y^1, x^2, y^2, \dots x^n, y^n)$. Let $\bar{M} = R \times_f C^{2n}$ be the warped product between the real line R and C^{2n}, where warping function is e^t and t being the global coordinates in R, then $\bar{M}$ is a Kenmotsu manifold. If D_θ and $D_\perp$ are any integrable slant and totally real distribution with corresponding submanifolds N_θ and $N_\perp$, such that $\frac{\partial}{\partial t}$ is tangential to $D_\perp$, then $N_\perp \times_f N_\theta$ is the pseudo-slant warped product submanifold of Kenmotsu manifolds.

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